

# Analyzing the degree of conflict among belief functions

Weiru Liu

*School of Electronics, Electrical Engineering and Computer Science, Queen's University Belfast, Belfast, BT7 1NN, UK*

Received 15 March 2005; received in revised form 6 April 2006; accepted 17 May 2006

Available online 30 June 2006

This paper is dedicated to Philippe Smets who sadly passed away in November 2005

---

## Abstract

The study of alternative combination rules in DS theory when evidence is in conflict has emerged again recently as an interesting topic, especially in data/information fusion applications. These studies have mainly focused on investigating which alternative would be appropriate for which conflicting situation, under the assumption that a *conflict* is identified. The issue of detection (or identification) of conflict among evidence has been ignored. In this paper, we formally define when two basic belief assignments are in conflict. This definition deploys quantitative measures of both the mass of the combined belief assigned to the emptyset before normalization and the distance between betting commitments of beliefs. We argue that only when both measures are high, it is safe to say the evidence is in conflict. This definition can be served as a prerequisite for selecting appropriate combination rules.

© 2006 Elsevier B.V. All rights reserved.

*Keywords:* Dempster–Shafer theory; Dempster's combination rule; Conflicting beliefs; Betting commitments

---

## 1. Introduction

When the Dempster–Shafer theory of evidence (DS theory) first appeared as a mechanism to model and reason with uncertain information in intelligent systems, criticisms on the counterintuitive results of applying Dempster's combination rule to conflicting beliefs soon emerged (e.g., [18,28,29]) where an almost impossible choice (with a very lower degree of belief) by both sources came up as the most possible outcome (with a very high degree of belief).

Since then, alternative combination rules have been explored to recommend where the mass of the conflicting belief from the two sources should land (e.g., [4,19,26]). Recently, due to the increasing applications of DS theory in intelligent fusion processes, Dempster's combination rule and its alternatives have been under the microscope again (e.g., [10,13,14,21]).

In [13], the three well-known alternatives, i.e., Smets's unnormalized combination rule (known as the conjunctive combination rule) [19], Dubois and Prade's disjunctive combination rule [4], and Yager's combination rule, are examined and a general combination framework is proposed. This new framework has a component that re-distributes the mass of the combined belief assigned to the emptyset (the false assumption) in a flexible way that it specifies which subsets can share this mass and by what proportions. In this way, the three alternatives listed above can all be subsumed by the framework.

---

*E-mail address:* [w.liu@qub.ac.uk](mailto:w.liu@qub.ac.uk) (W. Liu).

In [14], the weighted average is recommended in certain situations, such as, one piece of evidence contradicts with several other pieces of evidence which are consistent, to preserve the opinion from majority sources.

While the above two papers are still circling around the well-known alternatives, the method proposed in [10] takes a different approach. A new operator called *the consensus operator* is proposed which reduces a basic belief assignment on a set of values into a basic belief assignment on a binary set (a set with only two values), the combination is then carried out on this binary set. Three examples, two are commonly regarded as involving conflicting beliefs and one is with consistent beliefs, are examined in comparison with Dempster's rule of combination. It was concluded that the consensus operator could always produce a rational result for combining even conflicting beliefs. However, since this method always focuses on two elements when performing combinations, it may not be suitable for many complex situations where multiple values (not just two) should be preserved.

Furthermore, the consensus operator approach does not provide any indication whether the evidence to be combined may be conflicting. As it was concluded by the author "by looking at the result only, it does not tell whether the original beliefs were in harmony or in conflict" [10].

To justify whether original beliefs are in conflict has a big impact on selecting alternative combination rules [21]. So far there are no general mechanisms to measure the degree of conflict other than using the mass of the combined belief assigned to the emptyset before normalization, i.e.,  $m_{\oplus}(\emptyset)$  (see its definition in Section 2). In this paper, we mainly focus on this rather *ignored* topic. We study quantitatively when two sources can be defined as *conflict*. We argue that the conventional explanation that *a high mass value of the combined belief assigned to the emptyset before normalization indicates a conflict among the original beliefs* may not always be accurate. For example, if we have two distinct and totally reliable sources providing two basic belief assignments  $m_1$  and  $m_2$  as  $m_1(s_i) = m_2(s_i) = 0.2$  ( $i = 1, 2, \dots, 5$ ), then combining these basic belief assignments with Dempster's rule yields the mass being assigned to the emptyset as 0.8 before normalization. Under the current convention, this amount of mass would warrant a verdict that *these two pieces of evidence are in conflict and Dempster's rule should not be used*. In fact, these two pieces of evidence are consistent and using Dempster's rule produces a new basic belief assignment which is identical to either of them.

In order to avoid a wrong claim made by using only  $m_{\oplus}(\emptyset)$ , we propose an alternative method to measure the conflict among beliefs using a pair of values, the *mass of the combined belief allocated to the emptyset before normalization* and the *distance between betting commitments*. We also investigate the effect of these measures on deciding when Dempster's rule can be applied. We believe that this result is significant given that it decides subsequently whether Dempster's rule is appropriate and if not, what other rule should be considered.

The rest of the paper is organized as follows. In Section 2, we review the basic definitions in DS theory. In Section 3, we first examine the commonly accepted convention that  $m_{\oplus}(\emptyset)$  reveals the degree of conflict among two pieces of evidence and the deficiency associated with this convention. We then investigate the meaning of pignistic transformation and define the *distance between betting commitments* from two pignistic transformations. A formal definition consisting of two measures is proposed to judge when two pieces of evidence are *in conflict*. We also demonstrate that the distance between betting commitments is consistent with the distance between two pieces of evidence measured by the method in [11]. In Section 4, we explore the properties and behaviour of this pair of measures. In Section 5, we look at the impact of this new definition on the decision of whether Dempster's rule should be used in general and then discuss in particular the cases which show the difference in the decisions when using only  $m_{\oplus}(\emptyset)$  and using both of the measures. Section 6 concludes the main contribution of the paper and discusses the difference between conflict analysis and independence analysis among basic belief assignments.

## 2. Basics of the Dempster–Shafer theory

We review a few concepts commonly used in the Dempster–Shafer theory of evidence, as well as the conjunctive combination rule [21] and the disjunctive combination rule [4]. Let  $\Omega$  be a finite set called the frame of discernment.

**Definition 1.** [20] A basic belief assignment (bba) is a mapping  $m : 2^{\Omega} \rightarrow [0, 1]$  that satisfies  $\sum_{A \subseteq \Omega} m(A) = 1$ .

In Shafer's original definition which he called *the basic probability assignment* [22], condition  $m(\emptyset) = 0$  is required in Definition 1. Recently, some of the papers on Dempster–Shafer theory, especially since the establishment

of the Transferable Belief Model (TBM) [16], condition  $m(\emptyset) = 0$  is often omitted. A bba with  $m(\emptyset) = 0$  is called a normalized bba and is also known as a *mass function*.

**Definition 2.** The belief function from a bba  $m$  is defined as  $bel: 2^\Omega \rightarrow [0, 1]$ ,

$$bel(A) = \sum_{B \subseteq A} m(B).$$

When  $m(A) > 0$ ,  $A$  is called a focal element of the belief function.

**Definition 3.** [21] A categorical belief function is a belief function where its corresponding bba  $m$  satisfies

$$m(B) = \begin{cases} 1 & \text{if } B = A, \text{ where } A \neq \emptyset, A \neq \Omega, B \subseteq \Omega, \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

A categorical belief function has only one focal element and this element is neither empty nor the whole frame. In other words, a categorical belief function assigns the total belief to a single proper subset of the frame.

**Definition 4.** Let  $m_1$  and  $m_2$  be two bbas defined on frame  $\Omega$  which are derived from two distinct sources. Let the combined bba be  $m_\oplus = m_1 \oplus m_2$  by Dempster’s rule of combination where  $\oplus$  represents the operator of combination. Then

$$m_\oplus(A) = \frac{\sum_{B,C \subseteq \Omega, B \cap C = A} m_1(B)m_2(C)}{1 - \sum_{B,C \subseteq \Omega, B \cap C = \emptyset} m_1(B)m_2(C)}, \quad \forall A \subseteq \Omega, A \neq \emptyset$$

when  $\sum_{B,C \subseteq \Omega, B \cap C = \emptyset} m_1(B)m_2(C) \neq 1$ .

$\sum_{B,C \subseteq \Omega, B \cap C = \emptyset} m_1(B)m_2(C)$  is the mass of the combined belief assigned to the emptyset before normalization and we denote it as  $m_\oplus(\emptyset)$ . In the following, whenever we use  $m_\oplus(\emptyset)$ , we always associate it with this explanation unless otherwise explicitly stated.

The above rule is meaningful only when  $m_\oplus(\emptyset) \neq 1$ , otherwise, the rule cannot be applied.

**Definition 5.** [21] Let  $m_1$  and  $m_2$  be two bbas defined on frame  $\Omega$ . Their conjunctive combination, denoted as  $m_\odot = m_1 \odot m_2$ , is a new bba on  $\Omega$  defined as:

$$m_\odot(A) = \sum_{B,C \subseteq \Omega, B \cap C = A} m_1(B)m_2(C), \quad \forall A \subseteq \Omega,$$

where  $\odot$  represents the operator of combination.

**Definition 6.** [4] Let  $m_1$  and  $m_2$  be two bbas defined on frame  $\Omega$ . Their disjunctive combination, denoted as  $m_\oslash = m_1 \oslash m_2$ , is a new bba on  $\Omega$  defined as:

$$m_\oslash(A) = \sum_{B,C \subseteq \Omega, B \cup C = A} m_1(B)m_2(C), \quad \forall A \subseteq \Omega$$

where  $\oslash$  represents the operator of combination.

The conjunctive combination rule is applied when it is assumed that the two (distinct) pieces of evidence are both reliable. When only one of the two is fully reliable, but it is not clear which source it is, the disjunctive combination rule is recommended.

**Definition 7.** [20] Let  $m$  be a bba on  $\Omega$ . Its associated pignistic probability function  $BetP_m: \Omega \rightarrow [0, 1]$  is defined as

$$BetP_m(\omega) = \sum_{A \subseteq \Omega, \omega \in A} \frac{1}{|A|} \frac{m(A)}{1 - m(\emptyset)}, \quad m(\emptyset) \neq 1$$

where  $|A|$  is the cardinality of subset  $A$ .

$BetP_m$  can be extended as a function on  $2^\Omega$  as  $BetP_m(A) = \sum_{\omega \in A} BetP_m(\omega)$ . The transformation from  $m$  to  $BetP_m$  is called the *pignistic transformation*. When an initial bba gives  $m(\emptyset) = 0$ ,  $\frac{m(A)}{1-m(\emptyset)}$  is reduced to  $m(A)$ . In the following, we always assume that function  $BetP_m$  has been extended to  $2^\Omega$ .

$BetP_m(A)$  tells what is the total mass value that  $A$  can carry and it is referred to as the *betting commitment to A*. Function  $BetP_m$  is named the *probability expectation function* in [10].

### 3. Relationship between basic belief assignments

#### 3.1. $m_{\oplus}(\emptyset)$ versus conflict

So far we have used the word “conflict” without giving it an explicit explanation. In this subsection we investigate what it means when “two pieces of evidence (or two beliefs) are in conflict” in the context of Dempster–Shafer theory in the literature and how it is associated with  $m_{\oplus}(\emptyset)$ .

Let us first look at the following example.

**Example 1.** Let  $\Omega$  be a frame of discernment with four elements  $\{\omega_1, \omega_2, \omega_3, \omega_4\}$ . Assume two bbas, from two distinct sources, are defined as<sup>1</sup>

$$\begin{aligned} m_1(\{\omega_1, \omega_2\}) &= 0.9, & m_1(\{\omega_3\}) &= 0.1, & m_1(\{\omega_4\}) &= 0.0, \\ m_2(\{\omega_1, \omega_2\}) &= 0.0, & m_2(\{\omega_3\}) &= 0.1, & m_2(\{\omega_4\}) &= 0.9. \end{aligned}$$

Combining them using Dempster’s rule leads to a new bba  $m_{\oplus}$  with  $m_{\oplus}(\{\omega_3\}) = 1.0$  and  $m_{\oplus}(\emptyset) = 0.99$  before normalization.

Neither of the two strongly preferred choices by the two sources is preserved and the least preferred choice by the two sources is given the full credit after the combination with Dempster’s rule.

The first source is *almost* certain that the true hypothesis is either  $\omega_1$  or  $\omega_2$ , whilst the second source is *almost* certain that the true hypothesis is  $\omega_4$ . Since at most one and only one of these three hypotheses can be true, the beliefs from both sources cannot be held simultaneously. Hence these two beliefs largely contradict with each other. When this happens, it is said that the two beliefs are *in conflict*. Following this analysis, we give the following *qualitative* definition of conflict between two beliefs.

**Definition 8.** A conflict between two beliefs in DS theory can be interpreted qualitatively as one source strongly supports one hypothesis and the other strongly supports another hypothesis, and the two hypotheses are not compatible.

In Example 1, if we revise the two bbas as:

$$\begin{aligned} m'_1(\{\omega_1, \omega_2\}) &= 1.0, & m'_1(\{\omega_3\}) &= 0.0, & m'_1(\{\omega_4\}) &= 0.0, \\ m'_2(\{\omega_1, \omega_2\}) &= 0.0, & m'_2(\{\omega_3\}) &= 0.0, & m'_2(\{\omega_4\}) &= 1.0 \end{aligned}$$

then these two bbas *totally* contradict with each other. This is the situation when the *maximal conflict* occurs. Furthermore, the extreme case of maximal conflict appears when  $m_1(\omega_i) = 1$  for bba  $m_1$  and  $m_2(\omega_j) = 1$  for bba  $m_2$ , and  $\omega_i \neq \omega_j$ , where one source is absolutely sure that  $\omega_i$  is the right hypothesis and the other source is absolutely sure that  $\omega_j$  is the right hypothesis.

On the other hand, the *total absence of conflict* occurs when two bbas are identical. In this situation, whatever supported by one bba is equally supported by the other bba and there is no slightest difference in their beliefs. The extreme case of total absence of conflict is when  $m_1(A) = m_2(A) = 1$ , especially when  $|A| = 1$ .

**Example 2.** Let the first pair of bbas from two distinct sources on frame  $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\}$  be:

<sup>1</sup> This example shares the same spirit as the famous *Peter, Mary, Paul* murder suspect example. We include more elements in the frame to facilitate subsequent examples in the paper.

$$m_1(\{\omega_1\}) = 0.2, \quad m_1(\{\omega_2\}) = 0.2, \quad m_1(\{\omega_3\}) = 0.2, \quad m_1(\{\omega_4\}) = 0.2, \quad m_1(\{\omega_5\}) = 0.2,$$

$$m_2(\{\omega_1\}) = 0.2, \quad m_2(\{\omega_2\}) = 0.2, \quad m_2(\{\omega_3\}) = 0.2, \quad m_2(\{\omega_4\}) = 0.2, \quad m_2(\{\omega_5\}) = 0.2.$$

Dempster’s rule produces a combined bba, denoted as  $m_{\oplus}$ , which is identical to either of the two original ones and generates  $m_{\oplus}(\emptyset) = 0.8$  before normalization.

Now let the second pair of bbas from two distinct sources on the same frame be:

$$m_3(\{\omega_1\}) = 0.4, \quad m_3(\{\omega_1, \omega_2\}) = 0.4, \quad m_3(\{\omega_1, \omega_2, \omega_3\}) = 0.2,$$

$$m_4(\{\omega_1\}) = 0.4, \quad m_4(\{\omega_1, \omega_2\}) = 0.4, \quad m_4(\{\omega_1, \omega_2, \omega_3\}) = 0.2.$$

Combining these two bbas produces  $m'_{\oplus}(\emptyset) = 0.0$  for the combined bba  $m'_{\oplus}$ .

In this example, the two pairs of bbas have different characteristics. The bbas in the first pair are actually probability distributions and the corresponding belief functions are fully dissonant whilst the two bbas in the second pair have nested sets of focal elements and their corresponding belief functions are in fact necessity measures in possibility theory and are therefore fully consonant [2,9].

So far, in Dempster–Shafer theory, value  $m_{\oplus}(\emptyset)$  is commonly taken as the *quantitative* measure of this qualitative definition of conflict (e.g., [22]).

With regard to Example 1, qualitatively, the two beliefs are in conflict; quantitatively, the  $m_{\oplus}(\emptyset)$  value is very large. Therefore, the  $m_{\oplus}(\emptyset)$  value seems to accurately reflect the meaning of the qualitative analysis. Furthermore, Dempster’s rule indeed produces a result that is counterintuitive. As a consequence, there is no dispute about using  $m_{\oplus}(\emptyset)$  as a quantitative measure of conflict in this case and all other similar examples appeared in the literature.

However, if we look into Example 2, we see a different picture. First, if we follow the above convention in Dempster–Shafer theory and use the  $m_{\oplus}(\emptyset)$  value as the quantitative measure of conflict in beliefs, then the two bbas in the first pair could be classified as in conflict since  $m_{\oplus}(\emptyset) = 0.8$ . This conclusion is obviously wrong because we know already that these two bbas completely agree with each other although both bbas have lower confidence in every possible hypothesis. Second,  $m'_{\oplus}(\emptyset) = 0.0$  goes nicely with the fact that the two bbas in the second pair share totally the same beliefs. However, in general, it is very likely that  $m_{\oplus}(\emptyset) > 0$  holds for two identical bbas except when they have nested focal elements (like the second pair). Therefore,  $m_{\oplus}(\emptyset)$  values have no direct relationship with whether the two bbas under consideration are identical, i.e., absence of conflict.

In summary, it is not difficult to conclude from the above investigation that not all high  $m_{\oplus}(\emptyset)$  values indicate a conflict that can be satisfactorily interpreted qualitatively. As demonstrated by the first pair of bbas in Example 2, where the  $m_{\oplus}(\emptyset)$  value before normalization is high, but the two beliefs cannot be classified as in conflict qualitatively.

These examples prompt us to conclude that value  $m_{\oplus}(\emptyset)$  cannot be used as a quantitative measure of conflict between two beliefs, contrary to what has long been taken as a *fact* in the Dempster–Shafer theory community. We argue that  $m_{\oplus}(\emptyset)$  only represents the mass of *uncommitted belief* (or *falsely committed belief*) as a result of combination.

Since  $m_{\oplus}(\emptyset)$  itself is not sufficient as the quantitative measure of conflict between two beliefs, we must look into other criteria in order to reveal the relationship between two bbas. Let us recall the result of pignistic transformation  $BetP_m$ . If  $BetP_{m_1}$  and  $BetP_{m_2}$  are two pignistic probability functions for two bbas, then what does  $|BetP_{m_1}(A) - BetP_{m_2}(A)|$  reveal about  $A$ ?

### 3.2. Betting commitment versus conflict

**Example 3.** Let two bbas from two distinct sources on frame  $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$  be

$$m_1(\{\omega_1, \omega_2\}) = 0.8, \quad m_1(\{\omega_3\}) = 0.1, \quad m_1(\{\omega_4\}) = 0.1,$$

$$m_2(\{\omega_1, \omega_2\}) = 0.1, \quad m_2(\{\omega_3\}) = 0.1, \quad m_2(\{\omega_4\}) = 0.8.$$

Combining them with Dempster’s rule gives  $m_{\oplus}(\emptyset) = 0.83$  before normalization. Let  $A = \{\omega_4\}$  (where focal element  $A$  has the highest mass value in at least one bba), then the betting commitments to  $A$  from the two bbas are

$$BetP_{m_1}(A) = 0.1, \quad BetP_{m_2}(A) = 0.8$$

and the difference between them is

$$|BetP_{m_1}(A) - BetP_{m_2}(A)| = 0.7$$

which is high. It indicates that there is a significant difference from the two sources in believing that the true event is in  $A$ . At the same time, the mass of uncommitted belief after combination is also very high ( $m_{\oplus}(\emptyset) = 0.83$ ).

When both of these values are high, it could be the case that the large difference between betting commitments has contributed to the high amount of uncommitted belief ( $m_{\oplus}(\emptyset)$ ). However, these two measures can differ in other situations as demonstrated by the following example.

**Example 4.** Let  $m_1$  and  $m_2$  be two bbas defined in Example 2 (the first pair). Let  $A$  be a subset of the frame, say  $\{\omega_1\}$ , and  $BetP_{m_1}$  and  $BetP_{m_2}$  be the results of two pignistic transformations from  $m_1$  and  $m_2$  respectively, then

$$|BetP_{m_1}(A) - BetP_{m_2}(A)| = 0.2 - 0.2 = 0.0$$

which indicates that there is no difference between betting commitments to  $A$  from the two sources (and in fact to any subset).

In this example, two consistent beliefs result in no difference between betting commitments to any subset, whilst the mass of uncommitted belief after combination is still very high (0.8). Therefore, this amount of uncommitted belief cannot come from the difference between betting commitments but from other reasons, as a consequence, these two measures tell us two different aspects of the bbas involved.

**Definition 9.** Let  $m_1$  and  $m_2$  be two bbas on frame  $\Omega$  and let  $BetP_{m_1}$  and  $BetP_{m_2}$  be the results of two pignistic transformations from them respectively. Then

$$\text{difBetP}_{m_1}^{m_2} = \max_{A \subseteq \Omega} (|BetP_{m_1}(A) - BetP_{m_2}(A)|)$$

is called the *distance between betting commitments* of the two bbas.

Value  $(|BetP_{m_1}(A) - BetP_{m_2}(A)|)$  is the *difference between betting commitments to  $A$*  from the two sources. The distance of betting commitments is therefore the maximum extent of the differences between betting commitments to all the subsets.  $\text{difBetP}_{m_1}^{m_2}$  is simplified as  $\text{difBetP}$  when there is no confusion as which two bbas are being compared.

Obviously,  $\text{difBetP}_{m_1}^{m_2} = 0$  whenever  $m_1 = m_2$ , i.e., the distance between betting commitments is always 0 between any two identical bbas (total absence of conflict).

Given two bbas and their corresponding pignistic transformations, it is possible that these two bbas have the same betting commitment to a subset  $A$  (that is,  $BetP_{m_1}(A) = BetP_{m_2}(A)$ ), but have rather different betting commitments to another subset  $B$ . For this reason, we cannot use either min or mean to replace operator max in the above definition, since we want to find out the maximum, not the minimum or the average, level of differences between their betting commitments.

Notation  $\text{difBetP}$  coincides with the error measurement between a belief function and its approximation defined in [24], where  $BetP_{m_1}$  and  $BetP_{m_2}$  represent pignistic probability functions of a belief function and its approximation respectively.

### 3.3. Conflict between basic belief assignments

After analyzing the nature of both  $m_{\oplus}(\emptyset)$  and  $\text{difBetP}$ , we are now ready to quantitatively define when two bbas can be said in conflict.

**Definition 10.** Let  $m_1$  and  $m_2$  be two bbas. Let  $cf(m_1, m_2) = (m_{\oplus}(\emptyset), \text{difBetP})$  be a two-dimensional measure where  $m_{\oplus}(\emptyset)$  is the mass of uncommitted belief when combining  $m_1$  and  $m_2$  with Dempster's rule and  $\text{difBetP}$  be the distance between betting commitments in Definition 9.  $m_1$  and  $m_2$  are defined as *in conflict* iff both  $\text{difBetP} > \varepsilon$  and  $m_{\oplus}(\emptyset) > \varepsilon$  hold, where  $\varepsilon \in [0, 1]$  is the threshold of conflict tolerance.

Since there does not exist an “absolute meaningful threshold” of conflict tolerance satisfying all pairs of bbas [1], the choice of  $\varepsilon$  is largely subjective and application oriented. In general, the closer  $\varepsilon$  is to 1.0, the greater the conflict tolerance is. If we let  $\varepsilon$  be 0.8, then according to Definition 10 the bbas in Example 3 are in conflict. However, if we upgrade the threshold to 0.9, then the two bbas are not in conflict.

The need to use two measures instead of one in Definition 10 is further illustrated by the following example.

**Example 5.** Let  $m_1$  and  $m_2$  be two bbas from two distinct sources on  $\Omega = \{\omega_1, \dots, \omega_5\}$  as

$$m_1(\{\omega_1\}) = 0.8, \quad m_1(\{\omega_2, \omega_3, \omega_4, \omega_5\}) = 0.2, \quad \text{and} \\ m_2(\Omega) = 1.$$

Then the mass of uncommitted belief is 0.0, i.e.,  $m_{\oplus}(\emptyset) = 0.0$ , when  $m_1$  and  $m_2$  are combined with Dempster’s rule, which is traditionally explained as there is no conflict between the two bbas. Indeed, they do not contradict with each other as what is qualitatively defined in Definition 8. However,  $m_1$  is more committed whilst  $m_2$  is less sure about its preference. The difference in their opinions is reflected by  $\text{diffBetP} = 0.6$ . It says that the two sources have rather different beliefs as where the true hypothesis lies.

To summarize, a large distance between betting commitments alone does not guarantee that the beliefs under consideration can be qualitatively interpreted as in conflict, nor does a high  $m_{\oplus}(\emptyset)$  value itself. Only these two values together can serve this purpose.

### 3.4. Betting commitment versus a distance measure approach

In [11], a method for measuring the distance between bbas is proposed. This distance is defined as

$$d_{BPA}(m_1, m_2) = \sqrt{\frac{1}{2}(\vec{m}_1 - \vec{m}_2)^T \stackrel{D}{=} (\vec{m}_1 - \vec{m}_2)}$$

where  $\stackrel{D}{=}$  is a  $(2^\Omega \times 2^\Omega)$ -dimensional matrix with  $d[i, j] = |A \cap B|/|A \cup B|$ , and  $A \in 2^\Omega, B \in 2^\Omega$  are the names of columns and rows respectively (note, we define  $|\emptyset \cap \emptyset|/|\emptyset \cup \emptyset| = 0$ ). Given a bba  $m$  on frame  $\Omega$ ,  $\vec{m}$  is a  $2^\Omega$ -dimensional column vector (can also be called a  $2^\Omega \times 1$  matrix) with  $m_{A \in 2^\Omega}(A)$  as its  $2^\Omega$  coordinates.

$(\vec{m}_1 - \vec{m}_2)$  stands for vector subtraction and  $(\vec{m})^T$  is the transpose of vector (or matrix)  $\vec{m}$ . When  $\vec{m}$  is a  $2^\Omega$ -dimensional column vector,  $(\vec{m})^T$  is its  $2^\Omega$ -dimensional row vector with the same coordinates.  $((\vec{m})^T \stackrel{D}{=} \vec{m})$  thus is the result of normal matrix multiplications (twice).

For example, let  $\Omega = \{a, b\}$  be the frame and let  $m(\{a\}) = 0.6, m(\Omega) = 0.4$  be a bba on  $\Omega$ . Then

$$\vec{m} = \begin{bmatrix} 0.0 \\ 0.6 \\ 0.0 \\ 0.4 \end{bmatrix}$$

is a 4-dimensional column vector with row names  $(\emptyset, \{a\}, \{b\}, \Omega)$  and  $(\vec{m})^T = [0.0 \ 0.6 \ 0.0 \ 0.4]$  is the corresponding row vector with column names  $(\emptyset, \{a\}, \{b\}, \Omega)$ .  $\stackrel{D}{=}$  is thus a  $4 \times 4$  square matrix with  $(\emptyset, \{a\}, \{b\}, \Omega)$  as the names for both rows and columns.  $((\vec{m})^T \stackrel{D}{=} \vec{m})$  therefore means that matrix  $(\vec{m})^T$  is multiplied by  $\stackrel{D}{=}$  first to produce a new 4-dimensional row vector and then this row vector is multiplied by column vector  $\vec{m}$  to obtain a single numerical result.

In the examples provided by [11], among two bbas, one of them is always a categorical belief function, i.e., the total mass value is awarded to a proper subset of the whole frame. Most of the time, this subset contains only a single element.

We now take an example from the paper (Example 2, Fig. 6 in [11]) to see the relationship between the distance measure ( $d_{BPA}$ ) and the distance between betting commitments ( $\text{diffBetP}$ ).

**Example 6.** Let  $\Omega$  be a frame of discernment with 20 elements (or any number of elements that is pre-defined). We use 1, 2, etc. to denote element 1, element 2, etc. in the frame. The first bba,  $m_1$ , is defined as

$$m_1(\{2, 3, 4\}) = 0.05, \quad m_1(\{7\}) = 0.05, \quad m_1(\Omega) = 0.1, \quad m_1(A) = 0.8,$$

where  $A$  is a subset of  $\Omega$ . The second bba used in the example is

$$m_2(\{1, 2, 3, 4, 5\}) = 1.$$

There are 20 cases where subset  $A$  increments one more element at a time, starting from Case 1 with  $A = \{1\}$  and ending with Case 20 when  $A = \Omega$  as shown in Table 1. The comparison between  $d_{BPA}$  defined in [11] and the distance between betting commitments difBetP for  $m_1$  and  $m_2$  for these 20 cases is detailed in Table 1 and graphically illustrated in Fig. 1.

Table 1  
Comparison of  $d_{BPA}$ , difBetP, and  $m_{\oplus}(\emptyset)$  values of  $m_1$  and  $m_2$  when subset  $A$  changes. Here  $m_{\oplus} = m_1 \oplus m_2$

Cases	$d_{BPA}$	difBetP	$m_{\oplus}(\emptyset)$
$A = \{1\}$	0.7858	0.605	0.05
$A = \{1, 2\}$	0.6866	0.426	0.05
$A = \{1, 2, 3\}$	0.5633	0.248	0.05
$A = \{1, \dots, 4\}$	0.4286	0.125	0.05
$A = \{1, \dots, 5\}$	0.1322	0.125	0.05
$A = \{1, \dots, 6\}$	0.3883	0.258	0.05
$A = \{1, \dots, 7\}$	0.5029	0.355	0.05
$A = \{1, \dots, 8\}$	0.5705	0.425	0.05
$A = \{1, \dots, 9\}$	0.6187	0.480	0.05
$A = \{1, \dots, 10\}$	0.6553	0.525	0.05
$A = \{1, \dots, 11\}$	0.6844	0.560	0.05
$A = \{1, \dots, 12\}$	0.7081	0.591	0.05
$A = \{1, \dots, 13\}$	0.7274	0.617	0.05
$A = \{1, \dots, 14\}$	0.7444	0.639	0.05
$A = \{1, \dots, 15\}$	0.7592	0.658	0.05
$A = \{1, \dots, 16\}$	0.7658	0.675	0.05
$A = \{1, \dots, 17\}$	0.7839	0.689	0.05
$A = \{1, \dots, 18\}$	0.7944	0.702	0.05
$A = \{1, \dots, 19\}$	0.8042	0.714	0.05
$A = \{1, \dots, 20\}$	0.8123	0.725	0.05

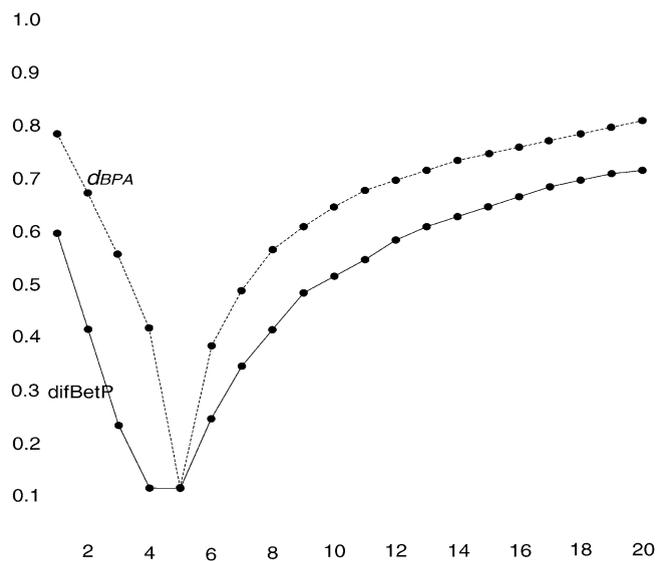


Fig. 1. Comparison of  $d_{BPA}$  and difBetP values of the two bbas detailed in Table 1 when subset  $A$  changes. The X-axis shows the sizes of subset  $A$  and the Y-axis gives the scale of  $d_{BPA}$  and difBetP.

Table 2  
Comparison of  $d_{BPA}$ , difBetP, and  $m_{\oplus}(\emptyset)$  values of the three pairs of bbas in Example 7

bbas	$d_{BPA}$	difBetP	$m_{\oplus}(\emptyset)$
$m_1^1, m_2^1$	<b>0.70</b>	0.700	0.83
$m_1^2, m_2^2$	0.574	0.464	0.19
$m_1^3, m_2^3$	<b>0.721</b>	0.600	0.00

From the figure we can see that both measures go up and down consistently, when the size of  $A$  changes. However, the difBetP values are always smaller than the corresponding  $d_{BPA}$  values, because the former selects the maximum value of the differences between betting commitments to all the subsets while the latter is the accumulated distance between two bbas.

The major drawback of  $d_{BPA}$  is its inability to distinguish between a large mass of uncommitted belief (i.e.,  $m_{\oplus}(\emptyset)$  is large, but difBetP is small) and a large difference between betting commitments (i.e.,  $m_{\oplus}(\emptyset)$  is small, but difBetP is large). This situation can be seen in the last few cases where the distance measure  $d_{BPA}$  between the two bbas is getting larger along with the growth of the size of  $A$ . Clearly the change in distances is caused by the difference between betting commitments not the amount of uncommitted belief. In fact, in all the cases value  $m_{\oplus}(\emptyset)$  remains unchanged. The following example makes the situation even more explicit.

**Example 7.** Let us consider three pairs of bbas on a frame with five elements:

$$\begin{aligned} \text{1st pair } \quad & m_1^1(\{\omega_1, \omega_2\}) = 0.8, \quad m_1^1(\{\omega_3\}) = 0.1, \quad m_1^1(\{\omega_4\}) = 0.1, \\ & m_2^1(\{\omega_1, \omega_2\}) = 0.1, \quad m_2^1(\{\omega_3\}) = 0.1, \quad m_2^1(\{\omega_4\}) = 0.8; \end{aligned}$$

$$\begin{aligned} \text{2nd pair } \quad & m_1^2(\{\omega_1, \omega_2, \omega_4\}) = 0.8, \quad m_1^2(\{\omega_3\}) = 0.1, \quad m_1^2(\{\omega_4\}) = 0.1, \\ & m_2^2(\{\omega_1, \omega_2\}) = 0.1, \quad m_2^2(\{\omega_3\}) = 0.1, \quad m_2^2(\{\omega_4\}) = 0.8; \end{aligned}$$

$$\begin{aligned} \text{3rd pair } \quad & m_1^3(\{\omega_1\}) = 0.8, \quad m_1^3(\{\omega_2, \omega_3, \omega_4, \omega_5\}) = 0.2, \\ & m_2^3(\Omega) = 1.0. \end{aligned}$$

The summary of  $d_{BPA}$ , difBetP, and  $m_{\oplus}(\emptyset)$  values of the three pairs is given in Table 2.

For the first pair the distance measure  $d_{BPA}$  between them reflects both the high values of difBetP and  $m_{\oplus}(\emptyset)$ , while the distance measure  $d_{BPA}$  in the third pair mainly reveals the difference between betting commitments of the pair, not the mass of uncommitted belief. By looking at these two  $d_{BPA}$  values alone, it is not possible to tell the difference between the first and the third pairs.

In summary, although the distance measure,  $d_{BPA}$ , between two bbas is a useful indicator as how different the belief distributions from the evidence are, however, this value itself cannot differentiate whether a pair of bbas are in conflict.

Since  $d_{BPA}$  and difBetP behave similarly, an interesting question is whether difBetP can be replaced by  $d_{BPA}$  in Definition 10 with a slightly lower threshold. We argue that this substitution, although looks feasible, offers a less accurate measure of the maximum degree of the difference between betting commitments (i.e., the degree of contradiction) of the two beliefs. Let us look at Example 7 again. The first pair of bbas are more contradictory in their beliefs (one strongly believes in  $\{\omega_1, \omega_2\}$  and another strongly believes in  $\{\omega_4\}$ ) than the two bbas in the third pair are. In the meantime, the difBetP value of the first pair is larger than that of the 3rd pair. That is, difBetP values are consistent with the severeness of the degree of contradiction in beliefs. In contrast, the  $d_{BPA}$  value for the first pair is smaller than that of the third pair, so these two values are not consistent with the severeness of the degree of contradiction in beliefs. Therefore, difBetP is a better measure than  $d_{BPA}$  for judging how contradict the two beliefs are.

#### 4. Properties and behaviour of the new measures

We deploy the examples (called cases) detailed in Table 3 to assist the explanation of the properties and behaviour of the two measures of  $cf(m_1, m_2)$  defined in Definition 10. We assume that two bbas,  $m_1$  and  $m_2$ , are from two distinct sources. We first examine their properties.

**Lemma 1.**  $cf(m_1, m_2) = \langle 1, 1 \rangle$  iff  $(\bigcup A_i) \cap (\bigcup B_j) = \emptyset$ , where  $A_i$  and  $B_j$  are the focal elements of the two belief functions with  $m_1$  and  $m_2$  as their corresponding bbas.

Lemma 1 says that when  $cf(m_1, m_2) = \langle 1, 1 \rangle$ , the two sources contradict completely in predicting what the true hypothesis is, i.e., the maximal conflict occurs, as shown by Case 10 in Table 3.

**Lemma 2.** If two bbas belong to one of the categories below then  $cf(m_1, m_2) = \langle 0.0, 0.0 \rangle$ : (a) identical categorical belief functions; (b) one of them is a vacuous belief function and the other is a probability function with evenly distributed probabilities; (c) two identical bbas with identical nested sets of focal elements.

Lemma 2 implies that when  $cf(m_1, m_2) = \langle 0.0, 0.0 \rangle$ , the two bbas do not have even slightest contradiction in predicting what the true hypothesis is, see Cases 11 and 12 of Table 3 for the first two situations.

**Lemma 3.**  $cf(m_0 \oplus m_1, m_0 \oplus m_2) \neq cf(m_1, m_2)$  for most bbas.

For example, if we let  $m_1$  and  $m_2$  be the two bbas as defined in Case 5 of Table 3, and  $m_0$  be a bba as

$$m_0(\{1, 2\}) = 0.8, \quad m_0(\{1, 3\}) = 0.2,$$

then  $cf(m_1, m_2) = \langle 0.0, \text{difBetP} \rangle$  whilst  $cf(m_0 \oplus m_1, m_0 \oplus m_2) = \langle m_{\oplus}(\emptyset) > 0.0, \text{difBetP} \rangle$ , where  $m_{\oplus}(\emptyset)$  is the mass of uncommitted belief after combining bbas obtained from  $m_0 \oplus m_1$  and  $m_0 \oplus m_2$  respectively.

All these lemmas can be proved easily.

Let us now describe the behaviour of  $cf(m_1, m_2)$ . Given two bbas,  $m_1$  and  $m_2$ , that are from distinct sources, there are four situations as how this pair of measures,  $cf(m_1, m_2) = \langle m_{\oplus}(\emptyset), \text{difBetP} \rangle$ , would reveal the relationship between the bbas.

**A:** when both  $m_{\oplus}(\emptyset)$  and  $\text{difBetP}$  have very low values, this pair of values indicate that there is little contradiction between  $m_1$  and  $m_2$ , although there exists a focal element  $A$  of  $m_1$  and  $B$  of  $m_2$  such that  $A \cap B = \emptyset$ , see Case 8 in Table 3. The extreme situation for this category is  $cf(m_1, m_2) = \langle 0.0, 0.0 \rangle$ .

**B:** when  $m_{\oplus}(\emptyset)$  has a relatively high value and  $\text{difBetP}$  has a relatively low value, this pair of values indicate that the two bbas have no apparent severe difference between betting commitments. However, most of the pairs of focal elements from the two bbas may have the empty intersection. It is also highly likely that none of the focal elements has a particularly high mass value, see Case 9 in Table 3. The extreme situation for this category is when  $\text{difBetP} = 0.0$  which suggests that there is no difference between their betting commitments, see Case 2 in Table 3.

**C:** when  $m_{\oplus}(\emptyset)$  has a relatively low value and  $\text{difBetP}$  has a relatively high value, this pair of values indicate a strong difference between betting commitments, but with little amount of mass of uncommitted belief. Therefore, the focal elements from the two belief functions that do not share any common elements (empty intersection) cannot both have large amounts of mass values, such as Case 7 in Table 3. The extreme situation for this category is when  $m_{\oplus}(\emptyset) = 0.0$ , such as Case 3 in Table 3.

**D:** both  $m_{\oplus}(\emptyset)$  and  $\text{difBetP}$  have high values. This pair of values reveal that the difference between betting commitments is high and the mass of uncommitted belief is also high. This is the situation where the two bbas may be said in conflict, subject to the actual value of the threshold of conflict tolerance, such as Case 6 in Table 3. The extreme situation for this category is  $cf(m_1, m_2) = \langle 1, 1 \rangle$ .

The graphical illustration of these situations is given in Fig. 2. The dark grey square within rectangle C in Fig. 2 shows those situations where  $m_{\oplus}(\emptyset)$  is small and which can be mistaken as that Dempster's rule is applicable. If Dempster's rule is applied, most of the masses of the combined belief will be assigned to some very small intersection subsets, since a high  $\text{difBetP}$  value means that many of the focal elements from the two bbas create either an empty intersection or a small intersection. The dotted square within rectangle B shows those situations where  $m_{\oplus}(\emptyset)$  is

Table 3

Twelve cases of two bbas on  $\Omega$  with  $|\Omega| = 10$ . We use 1, 2, etc. to denote element 1, element 2 etc. in  $\Omega$ . A pair of values  $(\{1\}, 0.8)$  (resp.  $(\{2\}, 0.2)$ ) in Case 1 under column name  $m_1$  means  $m_1(\{1\}) = 0.8$  (resp.  $m_1(\{2\}) = 0.2$ )

Case	$m_1$ :	$m_2$ :	$cf(m_1, m_2)$
1	$\{1\}, 0.8; \{2\}, 0.2$	$\{1\}, 0.8; \{2\}, 0.2$	$\langle 0.32, 0.0 \rangle$
2	$\{i\}, 0.1$	$\{i\}, 0.1$	$\langle 0.9, 0.0 \rangle$
3	$\{1, 2, 3, 4\}, 1.0$	$\{4, 5\}, 0.8; \{3, 6\}, 0.2$	$\langle 0.0, 0.65 \rangle$
4	$\Omega, 1.0$	$\{4, 5\}, 0.5; \{2\}, 0.5$	$\langle 0.0, 0.4 \rangle$
5	$\{1, 2, \dots\}, 0.8; \{2, 4\}, 0.2$	$\{1, 2, 3\}, 0.8; \{2, 3, 4\}, 0.2$	$\langle 0.0, 0.23 \rangle$
6	$m_1^1$ in Ex 7	$m_2^1$ in Ex 7	$\langle 0.83, 0.7 \rangle$
7	$m_1^2$ in Ex 7	$m_2^2$ in Ex 7	$\langle 0.19, 0.464 \rangle$
8	$\{1, 2\}, 0.8; \{1, 3\}, 0.2$	$\{1\}, 0.5; \{2\}, 0.1; \{1, 3\}, 0.4$	$\langle 0.02, 0.3 \rangle$
9	$\{1\}, 0.3; \{2\}, 0.3$ $\{3\}, 0.2; \{4\}, 0.2$	$\{1\}, 0.2; \{2\}, 0.2$ $\{3\}, 0.3; \{4\}, 0.3$	$\langle 0.76, 0.1 \rangle$
10	$\{1, 2\}, 1.0$	$\{3, 4, 5\}, 1.0$	$\langle 1.0, 1.0 \rangle$
11	$\{1, 2\}, 1.0$	$\{1, 2\}, 1.0$	$\langle 0.0, 0.0 \rangle$
12	$\{i\}, 1/ \Omega $	$\Omega, 1.0$	$\langle 0.0, 0.0 \rangle$

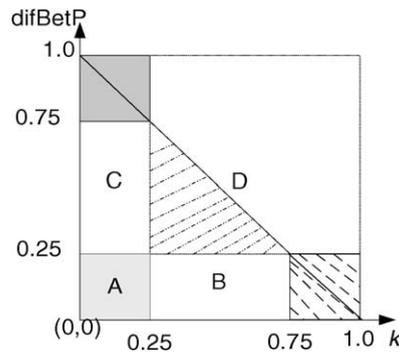


Fig. 2. An illustrative explanation of  $cf(m_1, m_2)$  where  $k = m_{\oplus}(\emptyset)$ . The two squares and two rectangles, denoted as A, D, B and C respectively, generated by the dotted lines starting with 0.25 on both axes correspond to the four situations above.

large and which traditionally suggests that Dempster’s rule cannot be applied. On the contrary, this square contains those cases where applying Dempster’s rule produces rational results, since the high  $m_{\oplus}(\emptyset)$  value is likely due to the low masses assigned to the focal elements, not the difference between betting commitments. The dotted triangle in square D shows some delicate situations. For example, when the  $(m_{\oplus}(\emptyset), difBetP)$  pair is closer to square A, it might still be feasible to apply Dempster’s rule, but when the pair is further away from square A, it becomes more problematic to apply the rule.

### 5. Impact on applying Dempster’s rule

There are several alternative combination rules to Dempster’s original version that aim at merging a pair of bbas when it is believed that Dempster’s rule is not adequate to use. In [21], Smets discussed in detail which rule should be used for what situation for both the categorical and general bba combinations. For example, when the mass of the combined belief  $m_{\oplus}(\emptyset) \neq 0.0$  and it is believed that one source is reliable and the other is not, the disjunctive rule [4] should be used. Alternatively, when there is no apparent reason for conflict, register the conflict with the conjunctive rule (the unnormalized Dempster’s rule) [21].

In summary, no matter for categorical or general bba combinations in [21] or in any other papers that investigate the alternative combination rules, as long as the combined  $m_{\oplus}(\emptyset) \neq 0.0$  (especially when this value is high), Dempster’s rule of combination is deemed to be inadequate. In this paper, we have demonstrated using various examples that this convention does not always hold. We argue that Dempster’s rule gives the most acceptable results in some of the situations where traditionally this rule is excluded from use on the basis that the  $m_{\oplus}(\emptyset)$  value is high. On the other hand, the perception that Dempster’s rule is applicable whenever  $m_{\oplus}(\emptyset) = 0.0$  is questioned when the application

of the rule narrows down the conclusion considerably. Therefore, we propose the following definition to recommend when Dempster's rule should be used and when it should not, given two bbas from two distinct sources.

**Definition 11.** Let  $cf(m_1, m_2) = \langle m_{\oplus}(\emptyset), \text{difBetP} \rangle$  be the two-dimensional measure defined in Definition 10 for bbas  $m_1$  and  $m_2$  from two distinct sources. Dempster's rule of combination is *not applicable* when Condition 1 holds; Dempster's rule of combination is *advised not to be applied* when Condition 2 holds; Dempster's rule of combination should be *applied with caution* when Condition 3 holds; and Dempster's rule is *applicable* when Condition 4 holds

1.  $m_1$  and  $m_2$  are in conflict according to Definition 10 (such as when  $\varepsilon = 0.85$ ),
2.  $\text{difBetP} \geq \varepsilon_2$ ,
3.  $\text{difBetP} \in (\varepsilon_1, \varepsilon_2)$ ,
4.  $\text{difBetP} \leq \varepsilon_1$ .

Here threshold  $\varepsilon_1 \in [0, 1]$  is assumed sufficiently small, such as 0.3 and threshold  $\varepsilon_2 \in [0, 1]$  is assumed sufficiently large, such as 0.8.

All these thresholds are subject to individual application requirements and indeed there is a very fine line (or a grey area) between Conditions 2 and 3. We will discuss the implications of these threshold values in Section 5.3. Condition 1 covers typical cases where a counterintuitive result will be generated, if the rule is used, as commonly accepted. The 2nd condition says that the rule should not be applied since it may eliminate most of the possible hypotheses given in the original evidence, even though the combined result is not necessarily counterintuitive. An exception to this scenario is when one bba is a vacuous belief function and the other assigns a high proportion of its belief to a subset. In this case the combined result does not eliminate hypotheses that are strongly supported by the 2nd bba. Our definition above does not include this situation, as we believe that combining a vacuous belief function with another bba is less interesting in practice. It should be pointed out that although Condition 2 subsumes Condition 1 and it can be refined to exclude Condition 1, i.e., to include a condition on  $m_{\oplus}(\emptyset)$ , we feel that the current description of Condition 2 is simpler and Condition 1 should always be checked before Condition 2 is applied. The 3rd condition advises that the rule should be used cautiously in order not to eliminate the true hypothesis during combination, although the result may not appear counterintuitive. Finally, the 4th condition implies two situations: (1) when  $m_{\oplus}(\emptyset)$  is small, the rule is perfectly applicable, as it is understood and practiced with most of the applications of DS theory; (2) when  $m_{\oplus}(\emptyset)$  is large, the rule is still applicable, as contrary to what most of the literature concluded.

### 5.1. Categorical belief functions

Now we apply this rule selection principle to categorical belief functions first.

Let two bbas from two distinct sources be  $m_1$  and  $m_2$  where each has one focal element with certainty. Let the frame of discernment be  $\Omega = \{\omega_1, \omega_2, \dots, \omega_{10}\}$ . We consider 10 cases listed in Table 4. The "Rule selection" column indicates our conclusions as whether Dempster's rule should be used based on Definition 11, in particular, the bold-faced suggestions are against the commonly accepted protocol.

For Cases 1 and 2, it is clear from Definition 10 that the two bbas are in conflict, therefore, there is no ambiguity that Dempster's rule should not be applied.

For all the other cases, the common feature is that the mass of uncommitted belief gives  $m_{\oplus}(\emptyset) = 0.0$ . Therefore, the default combination rule is Dempster's and the combined belief is assigned to the intersection of the two focal elements from the two bbas, regardless the size of each focal element as well as the size of the interaction. However, the distance between betting commitments gives a range of values which are almost in the reverse order of the ratio between the commonly supported subset (the intersection) and the union of the two focal elements, i.e.,  $\frac{|A \cap B|}{|A \cup B|}$ . When the distance between two betting commitments is low, such as Case 10, Dempster's rule is recommended to combine the bbas. However, when this value is high, such as Case 4, it is questionable whether the intersection of the two focal elements should still be the subset that carries all the combined belief, since these two bbas only agree on a small collection of elements.

Table 4

Ten cases of two categorical belief functions on  $\Omega$  with  $|\Omega| = 10$ . We use 1, 2, etc. to denote element 1, element 2 etc. in  $\Omega$ .  $m_1 : (A, 1)$  (resp.  $m_2 : (B, 1)$ ) means bba  $m_1$  (resp.  $m_2$ ) assigns mass value 1 to subset  $A$  (resp.  $B$ ). Case 1 and Case 2 etc. show the actual elements in  $A$  and  $B$  in different situations. The thresholds used are as defined in Definition 11 and the numerical number in the “Rule selection” column indicates which condition in Definition 11 is applied

Case	$m_1 : (A, 1)$	$m_2 : (B, 1)$	$\frac{ A \cap B }{ A \cup B }$	$cf(m_1, m_2)$	Rule selection
1	{3}	{5}	0	$\langle 1.0, 1.0 \rangle$	not Dempster’s (1)
2	{1, 2, 3}	{5, 6, 7, 8, 9}	0	$\langle 1.0, 1.0 \rangle$	not Dempster’s (1)
3	{3, 4}	{4, 5}	1/3	$\langle 0.0, 0.5 \rangle$	<b>caution with Dempster’s (3)</b>
4	{1, 2, 3, 4}	{4, 5, 6, 7, 8, 9}	1/9	$\langle 0.0, 0.83 \rangle$	<b>not Dempster’s (2)</b>
5	{1, 2, 3, 4}	{2, 3, 4, 5}	3/5	$\langle 0.0, 0.25 \rangle$	Dempster’s (4)
6	{1, 2, 3, 4}	$\Omega \setminus \{1\}$	3/10	$\langle 0.0, 0.67 \rangle$	<b>caution with Dempster’s (3)</b>
7	{3, 4, 5}	{4}	1/3	$\langle 0.0, 0.67 \rangle$	<b>caution with Dempster’s (3)</b>
8	$A$	$A$	1	$\langle 0.0, 0.0 \rangle$	Dempster’s (4)
9	$\Omega$	$\Omega$	1	$\langle 0.0, 0.0 \rangle$	Dempster’s (4)
10	{3, ..., 9}	{1, ..., 7}	5/9	$\langle 0.0, 0.286 \rangle$	Dempster’s (4)

Table 5

Twelve cases of two bbas on  $\Omega$  with  $|\Omega| = 10$ . We use 1, 2, etc. to denote element 1, element 2 etc. in  $\Omega$ . A pair of values ( $\{1\}, 0.8$ ) (resp.  $\{2\}, 0.2$ ) in Case 1 under column name  $m_1$  means  $m_1(\{1\}) = 0.8$  (resp.  $m_1(\{2\}) = 0.2$ ). The thresholds used are as defined in Definition 11 and the numerical number in the “Rule selection” column indicates which condition in Definition 11 is applied

Case	$m_1 :$	$m_2 :$	$cf(m_1, m_2)$	Rule selection
1	{1}, 0.8; {2}, 0.2	{1}, 0.8; {2}, 0.2	$\langle 0.32, 0.0 \rangle$	Dempster’s (4)
2	{ $i$ }, 0.1	{ $i$ }, 0.1	$\langle 0.9, 0.0 \rangle$	<b>Dempster’s (4)</b>
3	{1, 2, 3, 4}, 1.0	{4, 5}, 0.8; {3, 6}, 0.2	$\langle 0.0, 0.65 \rangle$	<b>caution with Dempster’s (3)</b>
4	$\Omega, 1.0$	{4, 5}, 0.5; {2}, 0.5	$\langle 0.0, 0.4 \rangle$	<b>caution with Dempster’s (3)</b>
5	{1, 2}, 0.8; {2, 4}, 0.2	{1, 2, 3}, 0.8; {2, 3, 4}, 0.2	$\langle 0.0, 0.23 \rangle$	Dempster’s (4)
6	$m_1^1$ in Example 7	$m_2^1$ in Example 7	$\langle 0.83, 0.7 \rangle$	<b>caution with Dempster’s (3)</b>
7	$m_1^2$ in Example 7	$m_2^2$ in Example 7	$\langle 0.19, 0.464 \rangle$	<b>caution with Dempster’s (3)</b>
8	{1, 2}, 0.8; {1, 3}, 0.2	{1}, 0.5; {2}, 0.1; {1, 3}, 0.4	$\langle 0.02, 0.3 \rangle$	Dempster’s (4)
9	{1}, 0.3; {2}, 0.3; {3}, 0.2; {4}, 0.2	{1}, 0.2; {2}, 0.2; {3}, 0.3; {4}, 0.3	$\langle 0.76, 0.1 \rangle$	<b>Dempster’s (4)</b>
10	{1, 2}, 1.0	{3, 4, 5}, 1.0	$\langle 1.0, 1.0 \rangle$	not Dempster’s (1)
11	{1, 2}, 1.0	{1, 2}, 1.0	$\langle 0.0, 0.0 \rangle$	Dempster’s (4)
12	{ $i$ }, $1/ \Omega $	$\Omega, 1.0$	$\langle 0.0, 0.0 \rangle$	Dempster’s (4)

### 5.2. General belief functions

Let us now re-examine the cases reported in Table 3 with Definition 11. Similar to Table 4, in Table 5 the bold-faced suggestions in the “Rule selection” column are based on Definition 11 and are in disagreement with traditional views of using Dempster’s rule.

We first look at Cases 2 and 9 in detail. Both cases work well with Dempster’s rule, so does the rule of average. However, other rules all fail. In Case 2, the mass of uncommitted belief ( $m_{\oplus}(\emptyset)$ ) is very high, yet this does not affect the use of Dempster’s rule. Case 9 would be another similar example where Dempster’s rule produces the same result as the average operator. Next, let us compare Cases 6 and 7. Based on Condition 3 of Definition 11, both cases can be

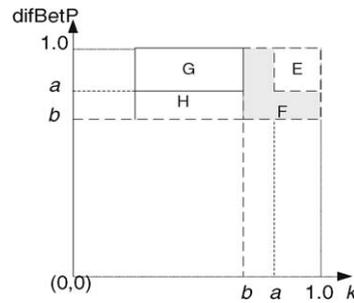


Fig. 3. Squares F and E are generated by the dotted lines starting with values  $b$  and  $a$  on both axes respectively, where  $k = m_{\oplus}(\emptyset)$ .

combined with Dempster's rule with caution. However, an interesting question is shall we be more cautious of Case 6 than of Case 7?

To summarize, a large  $m_{\oplus}(\emptyset)$  value itself does not rule out the possible application of Dempster's rule. On the contrary, a large  $m_{\oplus}(\emptyset)$  value coupled with a lower difBetP value strongly suggest the use of Dempster's rule.

### 5.3. Where the boundary lies

When defining *conflict* in Definition 10, we require that both of the values,  $m_{\oplus}(\emptyset)$  and difBetP, are greater than a threshold  $\varepsilon$ . An important question facing this is how large should  $\varepsilon$  be. We argue that the choice of  $\varepsilon$  is entirely subjective to the nature of an application. If an application can tolerate only a small amount of difference between two sources, then  $\varepsilon$  has to be a relatively smaller value such as 0.75 (square F in Fig. 3 shows the conflict situations). On the other hand, if it is possible to tolerate a great deal of difference between sources, then  $\varepsilon$  can be very close to 1.0 (square E in Fig. 3 shows the conflict situations). The former says that two sources have to be largely consistent to avoid being called in conflict, while the latter says that only when two sources provide highly contradictory information, they can be said in conflict.

Another interesting point with Definition 10 is whether we need one threshold for both measures or we need two different thresholds. Indeed, if two different thresholds are chosen, then it indicates that the measure that requires a smaller threshold plays a more important role than the other measure. For instance, when the threshold for difBetP is 1.0,  $cf(m_1, m_2)$  is reduced to the situation where only  $m_{\oplus}(\emptyset)$  is essential. Likewise, when the threshold for  $m_{\oplus}(\emptyset)$  is 1.0,  $cf(m_1, m_2)$  is reduced to the situation where only difBetP is useful. As we have demonstrated in examples earlier, neither of the two measures alone is capable of defining a conflict accurately. Therefore, both thresholds are essential to ensure that the two measures are used adequately.

Finally, it is a very fine line between when to caution the use of Dempster's rule and when not to use the rule, as shown between rectangles G and H in Fig. 3. To be more precise, when is difBetP too large so that not to use the rule and when is it large enough to caution the rule? The same question applies to the boundary between Conditions 3 and 4 in Definition 11. It is assumed these thresholds are fuzzy linguistic terms, just like *how tall is tall* in terms of a person's height, and there will be no universally agreed thresholds to satisfy all the applications.

## 6. Conclusions

Analyzing the relationship between a pair of bbas is important given that different combination rules can produce rather different results. A common perception of rejecting Dempster's rule is when the mass of uncommitted belief  $m_{\oplus}(\emptyset)$  is large and this value is widely taken as the reflection of *conflict* between the given beliefs. In this paper, we have investigated what really constitutes a conflict among two bbas. We examine not only the  $m_{\oplus}(\emptyset)$  value but also the distance between betting commitments, and conclude that only this pair of values together can reveal a real conflict. Neither  $m_{\oplus}(\emptyset)$  nor difBetP taken alone is adequate to precisely describe a conflict between two bbas. Whether there exists a single numerical measure, or another measure that can replace difBetP, to serve the same purpose as that of  $cf(m_1, m_2)$  remains to be an interesting question which deserves to be further investigated.

We also proposed a set of rules guiding whether Dempster's rule can be used, should be cautioned, is advised not to be used, or cannot be applied at all. The rejection of Dempster's combination rule on the ground that  $m_{\oplus}(\emptyset)$  is

large alone is no longer valid. Instead, this guideline can be applied along with suitability selected thresholds to tailor a particular application. When Dempster's rule is not applicable, the analysis of what has caused this and what other rule can substitute it is thoroughly discussed in [21].

In the past, apart from using  $m_{\oplus}(\emptyset)$  as the quantitative measure of conflict among two bbas, some other methods were proposed to measure the *conflict* or *confusion* or *non-specificity* in a single belief function, such as [3,5,8,12,17,25]. Since these approaches are designed for a single belief function only, they cannot be used to measure the conflict among pairs of belief functions in general and hence are not comparable with our method. Nevertheless, one may be attempted to apply these measures to the combined and unnormalized bba after combining two bbas with Dempster's rule. However, great care should be taken when using these measures. First, some of these methods are not applicable when  $m_{\oplus}(\emptyset) > 0$  (e.g., [8,25]), so adaptations of the methods are needed. Second, how should we interpret the value calculated with such a method, that is, are we examining this value on its own or are we comparing it with the results of using the same method to the two original bbas? These questions need to be fully investigated before these measures should be applied.

Another issue often surrounding the applicability of Dempster's rule is whether the two bbas to be combined are independent. There have been many papers discussing this, among them notably [27], on the justification of applying Dempster's rule. These discussions rely on the construction of an underlying probabilistic model to justify if the two bbas are probabilistically independent. In a situation where there does not exist an underlying probability model or it is too complex to establish the underlying probability model, the judgement of independence of the bbas cannot be done. Furthermore, belief functions can be interpreted in two ways [6], one is to take them as generalized probabilities and another is to take them as representing evidence. For the latter case, evidence can be the result of an agent's epistemic beliefs which does not necessarily come from probabilistic information. Two identical bbas do not suggest automatically that they must be dependent.

Nevertheless, it is worth to point out the distinction between the independence of sources and the independence of observations. With two bbas  $m_1$  and  $m_2$  derived from two independent sources (e.g., readings from two distinct sensors), the conflict analysis method developed in this paper can tell if these two bbas are in conflict and therefore whether they should be combined with Dempster's combination rule or with an alternative rule. In contrast, when these two bbas are from the same source (e.g., two readings at different time points from the same sensor), the conflict analysis between them can indicate whether there is a sudden change of the situation being observed (e.g., the target). Two bbas of this nature should not be combined regardless if they are in conflict. When there is no (or little) conflict between the two bbas, there is no dramatic change in the situation under observation. However, if the conflict does exist, then either the situation has changed completely or the source (e.g., the sensor) fails to function properly. In either case, the cause of the change, that is, the cause of the conflict between the *old* and the *new* bbas should be investigated.

The focus of this paper is not on the independence of bbas, rather, it is on whether two bbas are in conflict, no matter whether one has the intension to apply Dempster's rule. When one does want to combine two bbas from distinct sources, the conflict analysis between them defined in the paper will help to determine if Dempster's rule is appropriate. Therefore, fundamentally, this paper discusses a totally different issue from the independence requirement in the Dempster–Shafer theory of evidence.

When bbas are taken as a method to model an agent's beliefs, a bba defined by an agent can also be viewed as a way of expressing the agent's preferences over choices, with respect to masses assigned to different hypotheses (focal elements in terms of DS theory). The larger the mass assigned to a hypothesis is, the more preferred the hypothesis is. Therefore, hypotheses can be ordered according to the degree that they are preferred. Under this context, when the degree of disagreement of preferences is high as defined in social choice theory, we may then be able to conclude that the agents' beliefs are in conflict. An interesting future work here is whether numerous definitions of the degree of disagreement of preferences in social choice theory can be deployed to measure the conflict among bbas, and if so, how effective these measures could be in comparison with the approach proposed in this paper.

Finally, we would like to mention briefly logic-based interpretations of belief functions which may give us some insight on what a conflict means. Several papers (e.g. [7,15,23]) have discussed logic-based interpretations of DS theory, we follow the modal logic interpretation in [7] here. In terms of modal logic, the degree of belief on a hypothesis  $A$  (e.g., a subset  $A$  of a frame) can be taken as the accumulated sum of probabilities on those possible worlds in which  $A$  is necessarily true (defined by the valuation function with the modal operators). Let us denote the set of possible worlds (interpretations) in which  $A$  is necessarily true as  $Mod(A)$ . Under this notation, two belief functions,

$bel_1$  and  $bel_2$ , are said in conflict when  $Mod(A) \cap Mod(B) = \emptyset$  holds, where both  $m_1(A)$  and  $m_2(B)$  for  $bel_1$  and  $bel_2$  respectively are very large (close to 1). In other words, two belief functions are in conflict when there does not even exist one possible world that simultaneously supports the two hypotheses that are believed strongly by the two belief functions.

## Acknowledgement

I would like to express my sincere gratitude to Philippe Smets with whom I had numerous e-mail discussions about what a conflict was in the context of DS theory, and I would like to thank him for his critical remarks on the earlier versions of the paper. I would also like to thank the anonymous reviewers for their constructive comments which have helped me to improve the paper considerably.

## References

- [1] A. Ayoun, Ph. Smets, Data association in multi-target detection using the transferable belief model, *International Journal of Intelligent Systems* 16 (10) (2001) 1167–1182.
- [2] D. Dubois, H. Prade, On several representations of an uncertain body of evidence, in: M. Gupta, E. Sanchez (Eds.), *Fuzzy Information and Decision Processes*, North-Holland, Amsterdam, 1982, pp. 167–181.
- [3] D. Dubois, H. Prade, A note on measures of specificity for fuzzy sets, *International Journal of General Systems* 10 (4) (1985) 279–283.
- [4] D. Dubois, H. Prade, Representation and combination of uncertainty with belief functions and possibility measures, *Computational Intelligence* 4 (1988) 244–264.
- [5] T. George, N. Pal, Quantification of conflict in Dempster–Shafer framework: a new approach, *International Journal of General Systems* 24 (4) (1996) 407–423.
- [6] J. Halpern, R. Fagin, Two views of belief: belief as generalized probability and belief as evidence, *Artificial Intelligence* 54 (1992) 275–317.
- [7] D. Harmance, G. Klir, Z. Wang, Modal logic interpretation of Dempster–Shafer theory: An infinite case, *International Journal of Approximate Reasoning* 14 (1996) 81–93.
- [8] U. Hohle, Entropy with respect to plausibility measures, in: *Proc. of the 12th IEEE International Symposium on Multiple-Valued Logics*, 1982, pp. 167–169.
- [9] A. Hunter, W. Liu, Fusion rules for merging uncertain information, *Information Fusion* 7 (1) (2006) 97–134.
- [10] A. Josang, The consensus operator for combining beliefs, *Artificial Intelligence* 141 (1–2) (2002) 157–170.
- [11] A.L. Jousselme, D. Grenier, E. Bosse, A new distance between two bodies of evidence, *Information Fusion* 2 (2001) 91–101.
- [12] G. Klir, A. Ramer, Uncertainty in the Dempster–Shafer theory: A critical re-examination, *International Journal of General Systems* 18 (2) (1990) 155–166.
- [13] E. Lefevre, O. Colot, P. Vannoorenberghe, Belief function combination and conflict management, *Information Fusion* 3 (2002) 149–162.
- [14] C. Murphy, Combining belief functions when evidence conflicts, *Decision Support Systems* 29 (2000) 1–9.
- [15] E. Ruspini, The logical foundations of evidential reasoning, Technical Note 408, SRI International, Menlo Park, CA, 1986.
- [16] Ph. Smets, K. Kennes, The transferable belief model, *Artificial Intelligence* 66 (2) (1994) 191–234.
- [17] Ph. Smets, Information content of an evidence, *International Journal of Man-Machine Studies* 19 (1) (1983) 33–43.
- [18] Ph. Smets, Belief functions, in: Ph. Smets, A. Mamdani, D. Dubois, H. Prade (Eds.), *Non-Standard Logics for Automated Reasoning*, 1988, pp. 253–286.
- [19] Ph. Smets, The combination of evidence in the transferable belief model, *IEEE Transactions on Pattern Analysis and Machine Intelligence* 12 (5) (1990) 447–458.
- [20] Ph. Smets, Decision making in the TBM: the necessity of the pignistic transformation, *International Journal of Approximate Reasoning* 38 (2004) 133–147.
- [21] Ph. Smets, Analyzing the combination of conflicting belief functions, Technical Report, IRIDIA, Universite Libre de Bruxelles, 2004.
- [22] G. Shafer, *A Mathematical Theory of Evidence*, Princeton University Press, Princeton, NJ, 1976.
- [23] E. Tsiporkova, V. Boeva, B. De Baets, Dempster–Shafer theory framed in modal logic, *International Journal of Approximate Reasoning* 21 (2) (1999) 157–175.
- [24] B. Tessem, Approximations for efficient computation in the theory of evidence, *Artificial Intelligence* 61 (1993) 315–329.
- [25] R.R. Yager, Entropy and specificity in a mathematical theory of evidence, *International Journal of General Systems* 9 (4) (1983) 249–260.
- [26] R.R. Yager, On the Dempster–Shafer framework and new combination rules, *Information Sciences* 41 (1987) 93–138.
- [27] F. Voorbraak, On the justification of Dempster’s rule of combinations, *Artificial Intelligence* 48 (1991) 171–197.
- [28] L. Zadeh, A mathematical theory of evidence (book review), *AI Magazine* 5 (3) (1984) 81–83.
- [29] L. Zadeh, A simple view of the Dempster–Shafer theory of evidence and its implication for the rule of combination, *AI Magazine* 7 (1986) 85–90.